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Distance monotonicity and a new characterization of hypercubes

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Abstract

In this paper, we are interested in some metric properties of graphs. In particular, we investigate distance monotonicity in graphs. Straightaway, we revisit the notion of distance monotonicity. We then introduce interval distance monotone graphs, graphs which are not distance monotone but whose intervals are distance monotone. Finally, we obtain a new characterization of hypercubes involving this notion. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Several classes of graphs are defined by specific properties of their induced subgraphs. Often, one has to investigate subgraph properties of a specified class of graphs for obtaining characterizations of an interesting subclass by adding suitable conditions to some of these properties. One can then try to describe the role this subclass plays when studying the graphs under consideration. In particular, the intervals play a very interesting role when studying some classes of graphs that contain hypercubes as a subclass. An interval of a graph is analogous to the notion of an interval on the real number line in the sense that it consists of all vertices which link the endpoints of the interval. A hypercube is nothing else other than the interval between two of its antipodal vertices, i.e. vertices at maximal distance. Furthermore, all its intervals induce subcubes (hypercubes of less dimension). Among the classes of graphs defined in the foregoing spirit, one should mention interval monotone graphs [3] (every interval is convex) and distance monotone graphs [1,2] (every interval is closed).

In this paper, we are interested in a class of graphs related to the latter one above. First, we revisit distance monotone graphs. Second, we introduce and study the

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notion of interval distance monotone graphs (every interval induces a distance monotone graph). Involving this notion, we obtain finally a new characterization of hypercubes.

2. Preliminaries

In this section, we recall the necessary concepts, notions and notations that we will use. In general, our terminology is in accordance with [3].

All our graphs are finite, undirected, without loops and multiple edges and connected. Let $G=(V,E)$ be a graph and let S be a set of vertices of G. Throughout this paper, we will not distinguish between the set S and the subgraph of G induced by S. The neighborhood $N_G(u)$ of a vertex u consists of all vertices adjacent to u and its cardinality, denoted by $d_G(u)$, designates the degree of u. The maximum and the minimum degrees in G will be denoted by $\Delta(G)$ and $\delta(G)$, respectively.

For two vertices u and v, $N(u,v)$ contains their common neighborhood, which is simply $N(u)\cap N(v)$. The distance in G between u and v denoted by $d_G(u,v)$, represents the length of a u,v-geodesic (i.e. a shortest path between u and v). The diameter of G, denoted as $D(G)$, is the maximum distance between any pair of two vertices. The interval $I_G(u,v)$ can be defined by

$$I_G(u,v)=\{w\in V \mid d_G(u,w)+d_G(w,v)=d_G(u,v)\}$$

and it consists of all vertices on u,v-geodesics. The length of $I_G(u,v)$ is equal to $d_G(u,v)$. If no confusion arises, we will use all the above notations without indicating the reference graph. For instance, we will write $I(u,v)$ instead of $I_G(u,v)$.

We denote by $N_k(u)$, the set of vertices at a distance k from u and by $N_k(u,v)=I(u,v)\ \cap\ N^k(u)$. For any triple of vertices u,v,w in G, let $I(u,v,w)=I(u,v)\cap I(v,w)\ \cap\ I(w,u)$. A vertex x is a median of u,v,w if $x\in I(u,v,w)$. A median graph is a graph in which every triple of vertices has a unique median.

A subgraph H of G is convex if, for any two vertices u and v of H, the interval $I_G(u,v)$ is contained in H [3]. A graph G is called interval monotone if all its intervals are convex. An interval $I(u,v)$ is closed, if for any vertex w in $V\setminus I(u,v)$, there exists a vertex w' in $I(u,v)$ such that $d(w,w')>d(u,v)$. If all intervals of a graph G are closed, then G is said to be distance monotone [2]. If $d(w,\bar{w})\geqslant d(u,v)$, for all u and v in S, $\bar{w}$ is the diametrical vertex of w in S. If any vertex of G has a unique diametrical vertex, then G is a diametrical graph. A graph G is geodesic if for any vertices u and v of G, there is a unique u,v-geodesic in G. An interval $I(u,v)$ is (weakly) spherical if for any vertex w in $I(u,v)$, there exists a (unique) vertex $\bar{w}$ in $I(u,v)$ with $d(w,\bar{w})=d(u,v)$. A graph is (weakly) spherical if all its intervals are (weakly) spherical.

Finally, recall that the cartesian product $G\square G'$ of two graphs $G=(V,E)$ and $G'=(V',E')$ has as its vertex-set VxV', where (u,u') and (v,v') are adjacent if and only if either $u=v$ and $u'v'\in E'$ or $uv\in E$ and $u'=v'$.

The hypercube of dimension n, denoted by Q_n, can be looked at as a cartesian product of n copies of K_2 (the graph on two vertices linked by an edge). A Hamming graph is a cartesian product of complete graphs (not necessarily of a same order). Distance monotone graphs were introduced and studied by Burosch et al. [1]. We recall below some of the nice properties they obtained.

Theorem 1 (Burosch et al. [1]). *Let G be a distance monotone graph. Then*

(1) *G is bipartite.*
(2) *If v, w_1, w_2, w_3 are different vertices of G such that w_1, w_2, w_3 are adjacent to v, then there is a vertex u in G adjacent to w_1 and w_2 but not to w_3.*
(3) *If $\delta(G)=1$, then G is isomorphic to a path, and if $\delta(G)=2$, then G is isomorphic to a cycle of even length.*
(4) *If $\Delta(G) \geqslant 3$, then G is both weakly spherical and diametrical.*
(5) *$V(G)=I(u,v)$ for every $u,v \in V(G)$ such that $d(u,v)=D(G)$.*

They also obtained the following elegant characterization of hypercubes.

Theorem 2 (Burosch et al. [1]). *Let G be a graph with minimum degree $\delta(G) \geqslant 3$. Then G is a hypercube if and only if G is distance monotone and interval monotone.*

One can easily deduce from (3) of Theorem 1 that if G is a distance monotone graph such that $\Delta(G) \geqslant 3$, then $\delta(G) \geqslant 3$. On the other hand, observe that if a graph G is distance monotone, then any of its convex induced subgraphs is distance monotone. But we can have a distance monotone graph containing induced subgraphs which are distance monotone but not convex. For instance, a cycle of even length $2k$ is distance monotone and any of its induced paths of length k is distance monotone but not convex. However, if we only consider the subgraphs of G induced by intervals, we have the following.

Proposition 3. *Let G be a distance monotone graph and let u and v be two vertices in G. Then, $I(u,v)$ is distance monotone if and only if it is convex.*

Proof. If $\delta(G) \leqslant 2$, then the result is obvious, since, in this case, G is either empty or it is either isomorphic to a path or to a cycle of even length (Theorem 1(3)). Let us assume that $\delta(G) \geqslant 3$, and recall that in this case, G is also weakly spherical and diametrical (Theorem 1(4)). Take any interval $I_G(u,v)=H$ in G and assume that H is distance monotone. Let x and y be two vertices in H. Since H is a weakly spherical interval in G, it contains a vertex x' such that

$$d_G(x,x')=d_G(u,v)=D(H).$$

We also know that if a graph H is distance monotone, then every interval between two diametrical vertices induces H (Theorem 1(5)). Hence $I_H(x,x')=H=I_G(u,v)$.

Let us show that $I_G(x,x')=I_G(u,v)$. To this end, let z be any vertex in $I_G(u,v)$. Then, z belongs to $I_H(x,x')$ and we have

$$d_G(x,x')=d_H(x,x')=d_H(x,z)+d_H(z,x')\geqslant d_G(x,z)+d_G(z,x')\geqslant d_G(x,x').$$

It follows that $d_G(x,z)+d_G(z,x')=d_G(x,x')$. Consequently, $z\in I_G(x,x')$ and thus $I_G(u,v)\subset I_G(x,x')$.

Now, assume that the other inclusion is not satisfied and let z be in $I_G(x,x')\backslash I_G(u,v)$. Since G is distance monotone, then the interval $I_G(u,v)$ is closed and contains a vertex z' such that $d_G(z,z')>d_G(u,v)$.

Moreover, we have already shown that $I_G(u,v)\subset I_G(x,x')$. It follows that $z'\in I_G(x,x')$ and $d_G(z,z')\leqslant d_G(x,x')=d_G(u,v)$, leading thus to a contradiction. So, $I_G(u,v)=I_G(x,x')$. Hence, $y\in I_G(x,x')$ and $I_G(x,y)\subset I_G(x,x')=I_G(u,v)$. Consequently, $I_G(u,v)$ is convex.

The converse is obvious. □

3. Interval distance monotonicity and a characterization of hypercubes

A simple connected graph G is said to be interval distance monotone if for any two vertices u and v in G, the interval $I(u,v)$ induces a distance monotone graph.

Clearly, there are interval distance monotone graphs which are not distance monotone. For instance, complete graphs and cycles of odd lengths are simple examples of such graphs.

More generally, geodesic graphs are interval distance monotone, since in such graphs, every interval is isomorphic to an induced path.

Furthermore, hypercubes are also interval distance monotone, since any interval also induces a hypercube, which is obviously distance monotone.

Other examples of interval distance monotone graphs are the so-called extended odd graphs (also called Laborde–Mulder graphs) E_k, $k\geqslant 2$ [3]. The vertex-set of E_k is $A\subset 1,\dots,2k-1;\ |A|\leqslant k-1$, two vertices being adjacent if and only if the cardinality of their symmetric difference is either equal to 1 or $2k-2$. The small extended odd graphs are the complete graph K_4 when $k=2$ and the Greenwood–Gleason graph when $k=3$. In fact, E_k may be obtained in either of the following ways:

- take Q_{2k-1} and identify every two diametrical vertices,
- take Q_{2k-2} and add new edges joining diametrical vertices.

Another example of interval distance monotone graphs consists of Hamming graphs, which are simply cartesian products of complete graphs.

Finally, both the graphs in Fig. 1 are interval distance monotone.

Note that the graph G_1 in Fig. 1 is interval distance monotone, but is not distance monotone, interval monotone, weakly spherical, interval regular, nor vertex-transitive. The graph G_2 is not in the least regular.

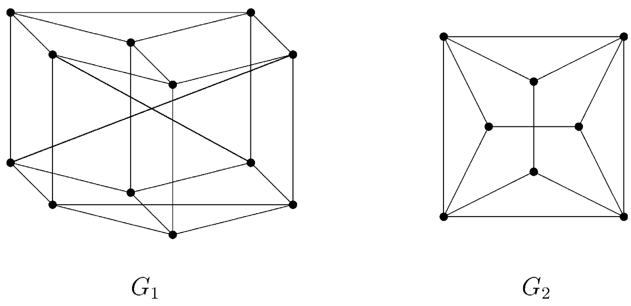

Fig. 1. Two interval distance monotone graphs.

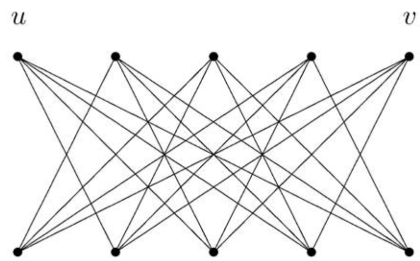

Fig. 2. $\tilde{K}_{5,5}$, a complete bipartite graph minus a perfect matching.

On the other hand, we can have distance monotone graphs which are not interval distance monotone. For instance, $\tilde{K}_{n,n}$ (the complete balanced bipartite graph minus a perfect matching), with $n \geqslant 5$, is distance monotone but is not interval distance monotone.

The graph of Fig. 2 is not interval distance monotone, since the interval $I(u,v)$ is isomorphic to $K_{2,3}$, and consequently, is not a distance monotone graph (by virtue of Theorem 1(3), since $\delta(K_{2,3}) = 2$). Accordingly, such a graph ($K_{2,3}$) cannot be contained in an interval distance monotone graph as an induced subgraph.

Using the notion of interval distance monotonicity, we can derive from Theorem 2 the following result.

Theorem 4. *Let G be a simple connected graph, with minimum degree $\delta(G) \geqslant 3$. Then G is isomorphic to a hypercube if and only if G is both distance monotone and interval distance monotone.*

Proof. Every hypercube is distance monotone and every interval in a hypercube induces a hypercube, and then is distance monotone. So, every hypercube is distance monotone as well as interval distance monotone.

Conversely, by Theorem 2, it suffices to show that a distance monotone and interval distance monotone graph is interval monotone. This follows immediately from Proposition 3. □

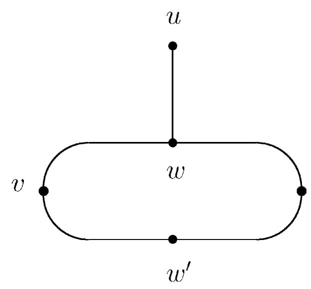

Fig. 3.

Proposition 5. *A graph G is a tree if and only if G is a bipartite interval distance monotone graph with* $\delta(G)=1$.

Proof. The "only if" part is obvious. For the "if" part, take any vertex v (different from u) in G, and let us denote by H the subgraph induced by $I(u,v)$. Since $\delta(H)=1$ and H is distance monotone, then H is isomorphic to a path.

Now, assume that G is bipartite but is not a tree. Let C be a shortest cycle in G with length $2k$, and let w be a nearest vertex of u in C. Finally, we denote by w', such a vertex in C, with $d(w,w')=k$ (Fig. 3).

Then $w\in I(u,w')$ and $C\subset I(u,w')$. Consequently, $I(u,w')$ is not a path, a contradiction.

Hence G is a tree. □

Proposition 6. *A graph G is isomorphic to an even cycle if and only if G is a bipartite interval distance monotone graph such that* $\delta(G)=2$.

Proof. The "only if" part is obvious. For the "if" part, it is clear that G contains a cycle. If a vertex of minimum degree v is contained in a cycle, let C be a shortest cycle (of length $2k$) in G containing v. If G is not a cycle, there is a vertex u not in C but having a neighbor w in C. Let us denote by w', the vertex in C with $d_G(w,w')=k$ (Fig. 3).

Since G is bipartite, then either u belongs to a w,w'-geodesic in G or w belongs to a u,w'-geodesic in G. In the first case, u lies in $I(w,w')=H$ and $d_H(w)\geqslant 3$. Then $\Delta(H)\geqslant 3$ and consequently, since H is distance monotone, $\delta(H)\geqslant 3$, a contradiction with $d_G(v)=2$. In the second case, w lies in $I(u,w')=H$ and $d_H(w)\geqslant 3$ and we are also done.

Now, let us assume that no vertex of minimum degree is contained in a cycle and let v be any vertex of minimum degree. Let C be a shortest cycle, with length $2k$, nearest to v and w, a nearest vertex of C to v. As above, let w' be the vertex in C with $d_G(w,w')=k$. Let $I(v,w')=H$. It is not difficult to see that w belongs to H and $d_H(w)\geqslant 3$ and we are done because of the same arguments above. □

Theorem 7. *Let G be a graph with minimum degree $\delta(G) \geqslant 3$. Then, G is a hypercube if and only if G is bipartite and interval distance monotone.*

Proof. The "only if" part is obvious. For the "if" part, let $G=(V,E)$ be a bipartite interval distance monotone graph with minimum degree $\delta(G)=3$. To prove that G is a hypercube, according to Theorem 4, we only have to show that G is distance monotone. For this, let us prove that if u and v are two diametrical vertices of G, then $I_G(u,v)=V$ (since in this case, the graph G will be distance monotone).

Let us denote by H, the graph induced by $I_G(u,v)$. Then H is distance monotone. Assume that there is a vertex w in $V \setminus I_G(u,v)$. Since G is connected, we can choose such a vertex w having a neighbor in $I_G(u,v)$.

Since G is bipartite, and $d_G(u,v)=D(G)$, the diameter of G, then $N(u) \cup N(v) \subset I(u,v)$ and consequently, w is neither adjacent to u nor to v. Now, let us show that w cannot be adjacent with any neighbor of u in $I_G(u,v)$. Let x be any neighbor of u (in $I_G(u,v)$). Since $N(u) \subset I_G(u,v)$, then $D(H) \geqslant 3$ and consequently, H is diametrical. Let $\bar{x}$ be in $I_G(u,v)$ such that $d_H(x,\bar{x})=D(H)$. Since H is distance monotone, $I_H(x,\bar{x})=I_G(u,v)$. Furthermore, $\bar{x}$ is adjacent to v. Let us show that $d_G(x,\bar{x})=d_H(x,\bar{x})$. Assume that $d_G(x,\bar{x})<d_H(x,\bar{x})$. Since G is bipartite, we cannot have $d_G(x,\bar{x})=d_H(x,\bar{x})-1$. If $d_G(x,\bar{x}) \leqslant d_H(x,\bar{x})-2$, then x and $\bar{x}$ must be in a u,v-geodesic in G. Consequently, every $x,\bar{x}$-geodesic must be contained in $I_H(x,\bar{x})$, leading to $d_H(x,\bar{x})<d_H(u,v)$, a contradiction. It follows that $I_H(x,\bar{x}) \subset I_G(x,\bar{x})$. Let us assume that there is a vertex z in $I_G(x,\bar{x}) \setminus I_H(x,\bar{x})$ and denote by K the subgraph of G induced by $I_G(x,\bar{x})$. Since K is distance monotone and $I_H(x,\bar{x}) \subset K$, there is a vertex $\bar{z}$ in K such that $d_K(z,\bar{z})>d_K(x,\bar{x})=D(K)$, a contradiction. Then necessarily $I_G(x,\bar{x})=I_G(u,v)$. Thus, since G is bipartite, $N(x) \subset I_G(x,\bar{x})$. Consequently, w cannot be adjacent to x.

Now, assume that for every pair of diametrical vertices $u,\bar{u}$ in G, w is not adjacent to any vertex in $N_k(u,\bar{u})$, and let us show that w cannot be adjacent to any vertex in $N_{k+1}(u,\bar{u})$.

To this end, assume that the contrary holds and let $y \in N_{k+1}(u,\bar{u}) \cap N(w)$. Let t be a neighbor of u in a u,y-geodesic of H, and $\bar{t}$ the diametrical vertex of t in H. One can easily see that $y \in N_k(u,\bar{u}) \cap N(w)$, a contradiction with the assumption.

Consequently, for any pair of diametrical vertices $u,\bar{u}$ in G, we have $I_G(u,\bar{u})=V$, and G is distance monotone. □

4. Concluding remarks and future directions

In this paper, we have used the notion of interval distance monotonicity in order to obtain a new characterization of hypercubes. Furthermore, we have obtained a complete characterization of bipartite interval monotone graphs (which are either trees or even cycles or hypercubes).

The class of interval distance monotone graphs also contains several well known and studied classes (Hamming graphs, extended odd graphs, etc.).

Problem 1. *How can all these classes of graphs be characterized using the notion of interval distance monotonicity?*

The second question we ask deals with the characterization of interval distance monotone graphs. Namely,

Problem 2. *Is it possible to obtain a characterization of interval distance monotone graphs in terms of forbidden induced subgraphs?*

It is not difficult to see that such a characterization is possible for graphs of diameter two. Indeed, a graph of diameter two is interval distance monotone if and only if it contains neither $K_{2,3}$ nor $K_{1,1,2}$ (a complete graph on four vertices minus an edge) as induced subgraphs.

It is also obvious that if every interval of a graph G is either isomorphic to a path or to a cycle or to a hypercube, then G is interval distance monotone.

Conjecture 1. A graph G is interval distance monotone if and only if each of its interval is either isomorphic to a path or to a cycle or to a hypercube.

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